Analysis 1, Summer 2024

List 3

Local extremes, concavity, infection points

68. (a) Calculate the derivative of $5x^2 - 3\sin(x)$.
    (b) Calculate the derivative of $10x - 3\cos(x)$.
    (c) Calculate the derivative of $10 + 3\sin(x)$.
    (d) Calculate the derivative of $3\cos(x)$.
    (e) Calculate the derivative of $-3\sin(x)$.

The **second derivative** of a function is the derivative of its derivative. The second derivative of $y = f(x)$ with respect to x can be written as any of

$$f''(x), \qquad f'', \qquad (f')', \qquad f^{(2)}, \qquad y'', \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

We say f is **twice-differentiable** if f'' exists on the entire domain of f. Higher derivatives (third, fourth, etc.) are defined and written similarly.

A twice-differentiable function $f(x)$ is **concave up** at $x = a$ if $f''(a) > 0$.
A twice-differentiable function $f(x)$ is **concave down** at $x = a$ if $f''(a) < 0$.

An **inflection point** is a point where the concavity of a function changes.

69. Compute the following second derivatives:
    (a) $f''(x)$ for $f(x) = x^{12}$
    (b) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = x^3 + x^8$
    (c) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ for $y = 8x - 4$
    (d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(5x^2 - 7x + 28)$
    (e) $f''(x)$ for $f(x) = -2x^8 + x^6 - x^3$
    (f) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = ax^2 + bx + c$

70. Find $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$ (the third derivative) for $f(x) = x^7$.

71. Give $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$ (the fifth derivative) for $f(x) = 5x^2 - 3\sin(x)$.

72. (a) Is the function $3x^2 + 8\cos(x)$ concave up or concave down at $x = 0$?
    (b) Is the function $3x^2 + 5\cos(x)$ concave up or concave down at $x = 0$?

73. On what interval(s) is $54x^2 - x^4$ concave up?

74. For each of the following functions, is $f''(0)$ is positive, zero, or negative?

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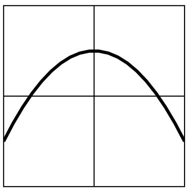

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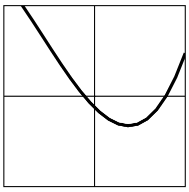

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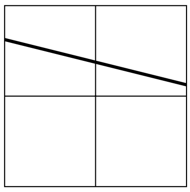

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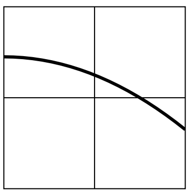

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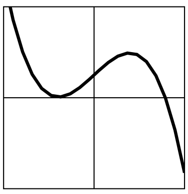

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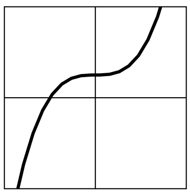

75. For $f(x) = x^3 - x^2 - x$,

(a) At what x value(s) does $f(x)$ change sign? That is, list values r where either $f(x) < 0$ when x is slightly less than r and $f(x) > 0$ when x is slightly more than r, or $f(x) > 0$ when x is slightly less than r and $f(x) < 0$ when x is slightly more than r.

(b) At what x value(s) does $f'(x)$ change sign?

(c) At what x value(s) does $f''(x)$ change sign?

(d) List all inflection points of $x^3 - x^2 - x$.

☆76. Give an example of a function with one local maximum and two local minimums but no inflection points.

77. Which graph below has $f'(0) = 1$ and $f''(0) = -1$?

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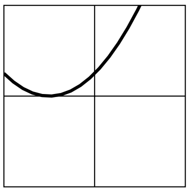

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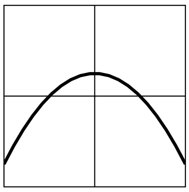

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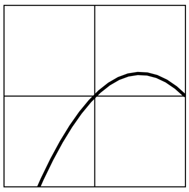

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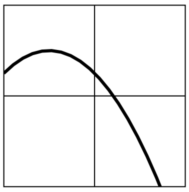

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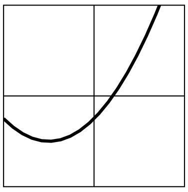

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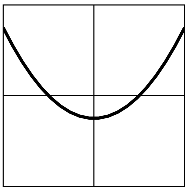

For a twice-differentiable function $f(x)$ with a critical point at $x = c$, ...

The Second Derivative Test:
- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ the test is inconclusive.

The First Derivative Test:
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then f has a local minimum at $x = c$.
- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then f has a local maxium at $x = c$.
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither a local minimum nor a local maximum.

78. Find all critical points of

$$4x^3 + 21x^2 - 24x + 19$$

and classify each as a local minimum, local maximum, or neither.

79. Find and classify[1] the critical points of

$$f(x) = x^4 - 4x^3 - 36x^2 + 18.$$

[1] "Classify the critical points" means to say whether each one is a local minimum, local maximum, or neither.

80. Find the inflection points of the function from Task 79.

☆81. Find and classify the critical points of $f(x) = x(6-x)^{2/3}$.

82. Find and classify the critical points of $\frac{3}{2}x^4 - 16x^3 + 63x^2 - 108x + 51$.

83. Label each of following statements as "true" or "false":
   (a) Every critical point of a differentiable function is also a local minimum.
   (b) Every local minimum of a differentiable function is also a critical point.
   (c) Every critical point of a differentiable function is also an inflection point.
   (d) Every inflection point of a differentiable function is also a critical point.

84. A twice-differentiable function $f(x)$ has the following properties:

$$f(4) = 2 \qquad f'(4) = 18 \qquad f''(4) = 0,$$
$$f(7) = 19 \qquad f'(7) = 0 \qquad f''(7) = -1.$$

   Label each of following statements as "true", "false", or "cannot be determined":
   (a) f has a critical point at $x = 4$.
   (b) f has a local maximum at $x = 4$.
   (c) f has an absolute maximum at $x = 4$.
   (d) f has an inflection point at $x = 4$.
   (e) f has a critical point at $x = 7$.
   (f) f has a local maximum at $x = 7$.
   (g) f has an absolute maximum at $x = 7$.
   (h) f has an inflection point at $x = 7$.

☆85. What is the maximum number of inflection points that a function of the form

$$\_\_x^6 + \_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$$

   can have?

Basic functions: $\frac{d}{dx}\left[x^p\right] = px^{p-1}$, $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$, $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$.	
Sum Rule: $(f+g)' = f' + g'$	**Product Rule:** $(f \cdot g)' = fg' + f'g$
Chain Rule: $(f(g))' = f'(g) \cdot g'$	**Quotient Rule:** $(f/g)' = \dfrac{gf' - fg'}{g^2}$

86. Give an equation for the tangent line to $y = \sin(\pi x)$ at $x = 2$.

87. Find the derivative of $\sin(\sqrt{\cos(2x^3 + 8)})$.

88. (a) Use the Quotient Rule to differentiate $\dfrac{\sin(x)}{x^4}$.
   (b) Use the Product Rule to differentiate $x^{-4}\sin(x)$.
   (c) Use algebra to compare your answers from parts (a) and (b).

89. At $x = 2$, is $\dfrac{x^2}{1+x^3}$ increasing, decreasing, or neither?

90. At $x = 0$, is $\sqrt{2 + \sin(x)}$ concave up, concave down, or neither?

91. Match the functions (a)-(d) with their derivatives (I)-(IV).

(a) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

(b) $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

(c) $\sec(x) = \dfrac{1}{\cos(x)}$

(d) $\csc(x) = \dfrac{1}{\sin(x)}$

(I) $\sec(x)\tan(x) = \dfrac{\sin(x)}{(\cos(x))^2}$

(II) $-(\csc(x))^2 = \dfrac{-1}{(\sin(x))^2}$

(III) $(\sec(x))^2 = \dfrac{1}{(\cos(x))^2}$

(IV) $-\csc(x)\cot(x) = \dfrac{-\cos(x)}{(\sin(x))^2}$

92. Match the functions (a)-(g) to their ***second*** derivatives (I)-(VII).

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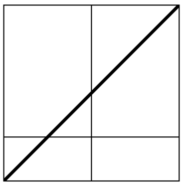

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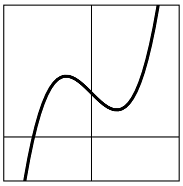

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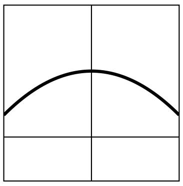

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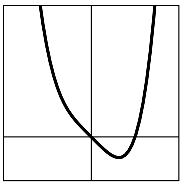

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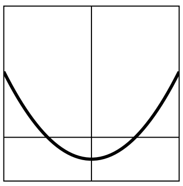

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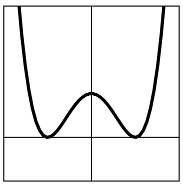

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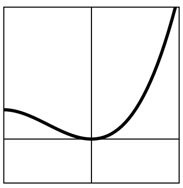

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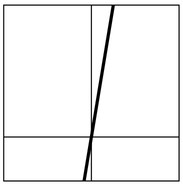

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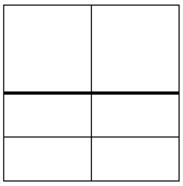

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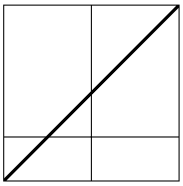

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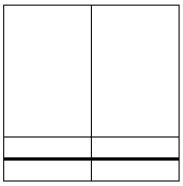

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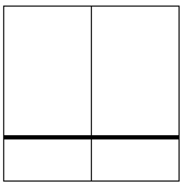

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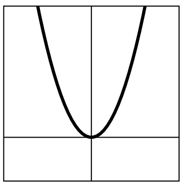

(VII)